HIGHER ORDER POLYNOMIALS

The two problems we will solve in this section are:
I. Given some zeros, find a function \( f(x) = Ax^3 + Bx^2 + Cx + D \) that has those zeros;
II. Given \( f(x) = Ax^3 + Bx^2 + Cx + D \), find the zeros of \( f(x) \).

HIGHER ORDER 1 CREATING POLYNOMIALS FROM REAL & COMPLEX ZEROS

Given some zeros, find a function \( f(x) = Ax^3 + Bx^2 + Cx + D \) that has those zeros:

Examples:
Zeros \(-1, 2, -3\) => \( f(x) = Ax^3 + Bx^2 + Cx + D \) where \( A=1, B=-2, C=1, D=-2; \) i.e. \( f(x) = x^3 + 2x^2 - 5x - 6 \).
Zeros \(0, 4, -3\) => \( f(x) = Ax^3 + Bx^2 + Cx + D \) where \( A=1, B=-2, C=1, D=-2; \) i.e. \( f(x) = x^3 + 2x^2 + x - 2 \).
Zeros \( \frac{1}{2}, 1, -2\) => \( f(x) = Ax^3 + Bx^2 + Cx + D \) where \( A=2, B=-1, C=-5, D=2; \) i.e. \( f(x) = 2x^3 + x^2 - 5x + 2 \).
Zeros \(-1, 0, 0\) (twice) => \( f(x) = Ax^3 + Bx^2 + Cx + D \) where \( A=1, B=1, C=D=0; \) i.e. \( f(x) = x^3 + x^2 \).
Zeros \(3, 1+i, 1-i\) => \( f(x) = Ax^3 + Bx^2 + Cx + D \) where \( A=1, B=-2, C=1, D=-2; \) i.e. \( f(x) = x^3 + 2x^2 + x - 2 \).
Zeros \(-1, 0, 0\) (twice) => \( f(x) = Ax^3 + Bx^2 + Cx + D \) where \( A=1, B=1, C=D=0; \) i.e. \( f(x) = x^3 + x^2 \).

HIGHER ORDER 2 SYNTHETIC DIVISION (S.D.)

Synthetic Division is a tool.

HIGHER ORDER 3 FIND ZEROS USING S.D.

Given \( f(x) = Ax^3 + Bx^2 + Cx + D \) and one of its zeros, find the other zeros of \( f(x) \).

In each case, give yourself one of the zeros (any one).
1) Find the zeros of \( f(x) = x^3 + 2x^2 - 5x - 6 \). Answer: \(-1, 2, -3\)
2) Find the zeros of \( f(x) = x^3 - x^2 - 12x \). Answer: \(0, 4, -3\).
3) Find the zeros of \( f(x) = x^3 + 2x^2 - 2 \). Answer: \(2, i, -i\).
4) Find the zeros of \( f(x) = 2x^3 + x^2 - 5x + 2 \). Answer: \(\frac{1}{2}, 1, -2\).
5) Find the zeros of \( f(x) = x^3 + x^2 \). Answer: \(-1, 0, 0\) (twice).
6) Find the zeros of \( f(x) = x^3 + 2x^2 + 8x - 6 \). Answer: \(3, 1+i, 1-i\).
7) Find the zeros of \( f(x) = x^3 - x^2 - 18 \). Answer: \(3, 1+i\sqrt{5}\).
8) Find the zeros of \( f(x) = x^3 - x + 6 \). Answer: \(-2, 1 \pm i\sqrt{2}\).
9) Find the zeros of \( f(x) = x^3 - 1 \). Answer: \(1, (-1 \pm i\sqrt{3})/2\).
10) Find the zeros of \( f(x) = x^3 - x^2 - 13x - 3 \). Answer: \(-3, 2\pm i\sqrt{5}\).
HIGHER ORDER 4 USE S.D. EVALUATE f(x) at x = k, i.e. Use S.D. to find f(k).

HIGHER ORDER 5 THE INTERMEDIATE VALUE THEOREM

HIGHER ORDER 6 FAR BEHAVIOR & Lead Term Test.
Use \((x^2-9)(x-2)=x^3-2x^2-9x+18\) to demo "genetic makeup" of \(f(x)\) and the battle between terms.

\[ \text{Lead Term Test (Ax}^N\text{ is the lead term)} \]

<table>
<thead>
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<th>(A)</th>
<th>(N) is even</th>
<th>(N) is odd</th>
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</thead>
<tbody>
<tr>
<td>(A &gt; 0)</td>
<td>ur, ul</td>
<td>ur, dl</td>
</tr>
<tr>
<td>(A &lt; 0)</td>
<td>dr, dl</td>
<td>dr, ul</td>
</tr>
</tbody>
</table>

HIGHER ORDER 7 THE RATIONAL ZEROS TEST
Here are a nice set of problems to have students work in pairs to illustrate various concepts:
List all possible Rational Zeros & Use SD to test them!
Have them work in pairs to Find Zeros and Sketch:
\[ f(x) = x^3 - 6x^2 + 11x - 6; 1, 2, 3 \]
\[ f(x) = 2x^3 + 3x^2 - 1; -1, -1, 1/2 \]
\[ f(x) = x^5 - x^4 - 3x^3 + 5x^2 - 2x = (x-1)^3(x+2)x \]
Read about other tests (desperate attempts) to find zeros - Descartes' rule of signs & upper and lower bounds theorem.

HIGHER ORDER 8 THE FUNDAMENTAL THEOREM OF ALGEBRA
Every polynomial has at least one complex zero.
Read page 331-332.

<table>
<thead>
<tr>
<th>(f(x))</th>
<th>Number of rational zeros</th>
<th>Number of irrational zeros</th>
<th>Number of complex zeros</th>
<th>Total number of zeros</th>
<th>Number of x-axis crossings</th>
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</thead>
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<tr>
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<td>(x^3-2x)</td>
<td></td>
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</tbody>
</table>

* Remember the Irrational Conjugate Zeros Theorem: If a polynomial with rational coefficients has a zero \(a+b\sqrt{c}\) where \(\sqrt{c}\) is irrational and \(a\) & \(b\) are rational, then \(a-b\sqrt{c}\) is also
a zero. HOWEVER, if the zero is of the form b√c then the conjugate theorem does not apply.

Example: f(x) = x³ - 2 has one irrational zero, \( \sqrt[3]{2} \), and two complex zeros \(-2^{\frac{2}{3}} \pm 2^{\frac{2}{3}} \sqrt[3]{3}i\).

HIGHER ORDER 9 PUTTING IT ALL TOGETHER
To sketch a polynomial, find the zeros and examine the far behavior:

Try These: Sketch using zeros and far behavior:

a) \( f(x) = (1/3)x^3(x-4)^2 \)

b) \( f = x^3 - 9x \)

c) \( y = x^4 - 4x^2 \)

d) \( x^3 - 4x^2 + 4x \)

Key Point: If a factor has an even exponent, that indicates double zeros and the graph bounces at that zero location!

KEY POINTS SUMMARY & REVIEW:
1. BIG 8 f notation/concepts
2. \( c_1f[c_2(x-c_3)]+c_4 \)
3. \( cf(x) \) does not affect zeros
4. Focus on \( x^2 \) and the \( x^2+1=0 \) lead in to Complex Numbers
5. zeros=>factors & factors=>zeros
6. Real zeros => x-incpts & Complex zeros => no x-incpt
7. We still use the QF tool even for polynomials of higher degree!
8. Multiplicity
9. Degree
10. Complex conjugate zeros: Complex conjugate zeros come in conjugate pairs if \( f(x) \) has real coefficients!
11. Irrational conjugate zeros: If \( f(x) \) has rational coefficients and has a zero \( a+b\sqrt{c} \)
    where \( \sqrt{c} \) is irrational and \( a \& b \) are rational, then \( a-b\sqrt{c} \) is also a zero; If the zero is of
    the form \( b\sqrt{c} \) then the irrational conjugate theorem does not apply. Example: \( f(x)=x^3-2 \)
    has one irrational zero, \( \sqrt[3]{2} \) (and two complex zeros \(-2^{\frac{2}{3}} \pm 2^{\frac{2}{3}} \sqrt[3]{3}i\)).
12. If \( (x-r)^N \) is a factor of \( f(x) \), then \( f \) bounces at \( x=r \) if \( N \) is even and \( f \) passes through the
    x-axis if \( N \) is odd.
13. The bigger \( N \) is in \( (x-r)^N \), the flatter \( f \) is near the zero \( r \).
14. Read the summary boxes in Lial on pp 321-348. These summarize the key points about
    synthetic division, conjugate pairs, zeros, factors, etc and how they are all related to
    each other.

A FEW REVIEW PROBLEMS:
find the zeros of \( x^3 + x^2 + 9x + 9 \) given 3i is a zero
find the zeros of \( 2x^4 + 5x^3 + 4x^2 + 5x + 2 \)
Find zeros of \( f(x)=1-x^3 \)
Find a poly with zeros 5 & $\sqrt{3}$
Find a poly with zeros 5 & $i$
If zeros are -1 & 0 & 0 (twice) then $f(x)=Ax^3+Bx^2+Cx+D$. Find the coefficients.