MAT 240 Calculus III  
Fall 2014 Quiz #5

NAME: Key

1. If \( \mathbf{r}(t) = \cos(3t) \mathbf{i} + t^2 \mathbf{j} + \sin(3t) \mathbf{k} \) describes the path of an object, find the following:

   A) The velocity of the object.
   \[
   \mathbf{r}'(t) = -3\sin(3t) \mathbf{i} + 2t \mathbf{j} + 3\cos(3t) \mathbf{k}
   \]

   B) The speed of the object.
   \[
   ||\mathbf{r}'(t)|| = \sqrt{9\sin^2(3t) + 4t^2 + 9\cos^2(3t)} = \sqrt{4t^2 + 9}
   \]

2. A sketch of the position vector \( \mathbf{r}(t) = \left( \frac{1}{3}t^3 + \frac{2}{3} \right) \mathbf{i} + (2t) \mathbf{j} \) is shown below. Do the following:

   A) Find the position vector, the velocity vector, and the acceleration vector when \( t = -2 \).
   \[
   \mathbf{r}(-2) = \langle -2, -4 \rangle \quad \mathbf{r}'(-2) = \langle 2, 4 \rangle \quad \mathbf{r}''(-2) = -4 \mathbf{i}
   \]

   B) On the graph provided, sketch the velocity vector and the acceleration vector at \( t = -2 \) such that the initial points of these two vectors are at the terminal point of the position vector.

3. Find the position vector given the following: \( \mathbf{v}(t) = e^{-t} \mathbf{i} + \sin(t) \mathbf{j} \), and \( \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} \)

   \[
   \mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \int e^{-t} \mathbf{i} \, dt + \int \sin(t) \mathbf{j} \, dt
   \]
   \[
   \mathbf{r}(t) = (-e^{-t} + C_1) \mathbf{i} + (-\cos(t) + C_2) \mathbf{j}
   \]
   \[
   \mathbf{r}(0) = (-1 + C_1) \mathbf{i} + (-1 + C_2) \mathbf{j}
   \]

4. Find a set of parametric equations for the tangent line to the curve given by \( \mathbf{r}(t) = \left( \frac{1}{16}t^2, \sqrt{t}, -\frac{1}{4}t^3 \right) \) at \( t = 4 \).

   \[
   \mathbf{r}'(t) = \left( \frac{1}{8}t, \frac{1}{2\sqrt{t}}, -\frac{3}{4}t^2 \right) \quad \mathbf{r}'(4) = \left( \frac{1}{8}(4), \frac{1}{2\sqrt{4}}, -\frac{3}{4}(4)^2 \right) = \left( \frac{1}{2}, \frac{1}{4}, -12 \right)
   \]

   \[
   x = \frac{1}{2} t + 1 \\
   y = \frac{1}{4} t + 2 \\
   z = -12t - 16
   \]
5. A ball rolls off (launch angle $0^\circ$) a 64 foot cliff at a constant speed of 8 ft/s, do the following:

A) Find a vector valued function for the path of the ball. The gravitational constant is 32 ft/s^2

$$\vec{r}(t) = (v_0 \cos\theta) t \hat{i} + \left( \frac{1}{2} g t^2 + v_0 \sin\theta t + h \right) \hat{j} = \left( 8 \cos 0 \right) t \hat{i} + \left( -16 t^2 + 16 \sin 0 t + 64 \right) \hat{j}$$

$$\vec{r}(t) = 8t \hat{i} + (-16t^2 + 64) \hat{j}$$

B) How long does it take for the ball to hit the floor after it leaves the table?

$$\vec{r}(t) = 0 \text{ in the } y \text{ direction.}$$

$$-16t^2 + 64 = 0 \Rightarrow t^2 = 4 \Rightarrow t = \pm 2$$

$$t = 2 \text{ seconds.}$$

C) How far horizontally did the ball travel when it hits the floor?

$$\vec{r}(t) = 8(t)$$

$$16 \text{ feet.}$$

6. If $r(t) = 2t^2 \hat{i} + 3t \hat{j}$

A) Find $T(1)$

$$\vec{r}'(t) = 4t \hat{i} + 3 \hat{j}$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{4t \hat{i} + 3 \hat{j}}{\sqrt{16t^2 + 9}}$$

$$T(1) = \frac{4 \hat{i} + 3 \hat{j}}{\sqrt{25}}$$

B) Find $a_T$ at $t = 1$

$$a_T = \frac{\vec{a} \cdot T}{\|T\|^2} = \frac{1}{\sqrt{25}} (4 \hat{i} + 3 \hat{j})$$

C) Find $a_N$

$$a_N = \frac{\vec{a} \cdot N}{\|N\|^2} = \frac{1}{25} (4 \hat{i} + 3 \hat{j})$$

7. If $r(t) = 2 \sin(t) \hat{i} + \sqrt{5} t \hat{j} + 2 \cos(t) \hat{k}$

A) Find $T(t)$

$$\vec{r}'(t) = \left< 2 \cos(t), \sqrt{5}, -2 \sin(t) \right>$$

$$\|\vec{r}'(t)\| = \sqrt{4 \cos^2(t) + 5 + 4 \sin^2(t)} = \sqrt{9} = 3$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{2 \cos(t) \hat{i} + \sqrt{5} \hat{j} - 2 \sin(t) \hat{k}}{3}$$

B) Find $N(t)$

$$\vec{r}'(t) = \left< -\frac{2}{3} \sin(t), 0, -\frac{2}{3} \cos(t) \right>$$

$$\|\vec{r}'(t)\| = \sqrt{\frac{4}{9} \sin^2(t) + \frac{4}{9} \cos^2(t)} = \frac{2}{3}$$

$$N(t) = \frac{1}{\frac{2}{3}} \left( -\frac{2}{3} \sin(t) \hat{i}, 0, -\frac{2}{3} \cos(t) \hat{k} \right) = \left< -\sin(t), 0, -\cos(t) \right>$$

C) Find $a_T$

$$a_T = \frac{\vec{a} \cdot T}{\|T\|^2} = \frac{1}{\frac{2}{3}} \left( \frac{2}{3} \hat{i} \right) = \frac{2}{3}$$

D) Find $a_N$

$$a_N = \frac{\vec{a} \cdot N}{\|N\|^2} = \frac{1}{\frac{2}{3}} \left( \frac{2}{3} \hat{i} \right) = \frac{2}{3}$$
8. An object has a \( T(2) = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \) and \( N(2) = \left( \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right) \). Furthermore, at the time value of \( t = 2 \), \( a_T = 5\sqrt{3} \) and \( a_N = 4\sqrt{14} \). Use this information to help you find the acceleration of the object at the time value of \( t = 2 \).

\[
\mathbf{a} = a_T T + a_N N = 5\sqrt{3} \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle + 4\sqrt{14} \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right\rangle = \left\langle 5, 5, 5 \right\rangle + \left\langle 2, 8, -12 \right\rangle = \left\langle 9, 13, -7 \right\rangle.
\]

9. Find the arclength of the curve given by \( r(t) = \frac{2}{3} t^{\frac{3}{2}} \mathbf{i} + t \mathbf{j} + 3t \mathbf{k} \) over the interval \([0, 6]\).

\[
\ell = \int_{0}^{6} \sqrt{\left\| r'(t) \right\|^2} \, dt = \int_{0}^{6} \sqrt{\left( t^{\frac{3}{2}} \right)^2 + 1^2 + 3^2} \, dt = \int_{0}^{6} \sqrt{t^3 + 10} \, dt.
\]

10. If \( f(x, y) = 3xy + 2y^2 \), do the three items below (simplify your answers):

A) \( f(-2, 3) \)

\[
2 = 3(-2)(3) + 2(3)^2 \quad 2 = -18 + 18 \quad z = 0
\]

B) \( f(x, 4x) \)

\[
2 = 3x(4x) + 2(4x)^2 \quad 2 = 12x^2 + 32x^2 \quad z = 44x^2
\]

C) \( f(x + \Delta x, y) - f(x, y) \)

\[
\frac{3(4 + \Delta x)^2 - 2\Delta x^2}{3} = \left[ 3xy + 3\Delta xy - 2\Delta x^2 - 3xy - 3\Delta xy \right] = 3\Delta x \frac{y}{y_0}
\]

11. Sketch the domain of the following function on the xy-plane provided. Be clear on whether or not the boundary of the domain is included or excluded from the domain.

\( f(x, y) = \sqrt{\frac{y - 1}{x - 1}} \)

\[
y \geq 2 \quad \text{x} \neq 1
\]

Boundary not included.
12. Sketch the domain of the following function on the xy-plane provided. Be clear on whether or not the boundary of the domain is included or excluded from the domain.

\[ f(x, y) = \frac{1}{\sqrt{4x^2 + 9y^2 - 36}} \]

\[ 4x^2 + 9y^2 - 36 \geq 0 \] Test:

\[ 4(0)^2 + 9(0)^2 = 0 \] \[ (0, 0) \]

\[ x^2 + \frac{y^2}{4} \neq 1 \]

**Included**

Boundary not included

13. If \( f(x, y) = y^2 - 4x^2 \) then do the following:

A) Sketch the level curve for \( z = 16 \)

\[ y^2 - 4x^2 = 16 \]

\[ \frac{y^2}{16} - \frac{x^2}{4} = 1 \]

B) Sketch the level curve for \( z = -16 \)

\[ \frac{y^2}{16} - \frac{x^2}{4} = -1 \]

13. A contour map of a function \( f(x, y) \) is shown below. Use it to sketch the 3D surface represented by \( f(x, y) \).