Section 4.6: The Counting Principle and Permutations
We introduced the Counting Principle earlier in the chapter.
Counting Principle: If a first experiment can be performed in M distinct ways and a second experiment can be performed in $N$ distinct ways then the two experiments in that specific order can be performed in $\mathrm{M} \times \mathrm{N}$ distinct ways.

The next three problems examples of the Counting Principle. They may be a little more involved, but the strategy to solve them is identical to what we have already done. To use the Counting Principle create a spot for each object that needs to be placed. Then we write the number of choices for each spot and multiply the numbers to get an answer.

Example: A computer password is to consist of two lower case letters followed by four digits. Determine how many passwords are possible if.....
A) Repetition of letters and digits is permitted

First create a spot for each object that needs to be placed.


Next place numbers in each spot and multiply the numbers.
Each of the spot marked " $L$ " will get a 26 above it as there are 26 letters. . I won't increment down by one as repetition is allowed.

Each of the spot marked " $\#$ " will get a 10 above it as there are 10 digits $(0,1,2,3,4,5,6,7,8,9)$ to choose from. I won't increment down by one as repetition is allowed.
$\underline{26} * \underline{26} * \underline{10} * \underline{10} * \underline{10} * \underline{10}=6,760,000$
L L \# \# \# \#

Answer: 6,760,000 passwords
B) Repetition of letters and digits is NOT permitted

First create a spot for each object that needs to be placed.

$$
\overline{\mathrm{L}} \overline{\mathrm{~L}} \overline{\#} \overline{\#} \overline{\#} \overline{\#}=
$$

Next place numbers in each spot and multiply the numbers.

Each of the spot marked "L" will get a 26 above it as there are 26 letters. I need to increment down by one as repetition is NOT allowed.

Each of the spot marked " $\#$ " will get a 10 above it as there are 10 digits $(0,1,2,3,4,5,6,7,8,9)$ to choose from. I need to increment down by one as repetition is NOT allowed.

$$
\frac{26}{L} * \frac{25}{L} * \frac{10}{\#} * \frac{9}{\#} * \frac{8}{\#} * \frac{7}{\#}=3,276,000
$$

Answer: 3,276,000 passwords
C) The first letter must be a vowel ( $\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}$ ) and repetition of letters is NOT permitted. The first digit cannot be a 0 , and repetition of digits is NOT permitted.

$$
\overline{\mathrm{V}} \overline{\mathrm{~L}} \frac{-}{\#} \overline{\#} \overline{\#} \frac{}{\#}=\quad \text { different passwords }
$$

The first letter has 5 choices as it needs to be a vowel.

The second letter has 25 choices. Repetition is not permitted. I can use any of the 25 letters not already used.

The first digit has 9 choices as it can be any of the 10 digits other than 0 . \{1,2,3,4,5,6,7,8,9\}

The second digit also has 9 choices as it can be the 0 and any of the 8 numbers not selected for the first number.

The third and $4^{\text {th }}$ digit will increment down 1 as repetition is NOT permitted.
$\underline{5} * \underline{25} * \underline{9} * \underline{9} * \underline{8} * \underline{7}=567,000$
V L \# \# \# \#

Answer: 567,000 passwords

Example: Bob, Sue, Larry, Sally and Fred are waiting in line to buy concert tickets.
A) In how many different ways can they stand in line?

I make 5 lines, one for each person that is going to wait in the line.
$\qquad$
This has to be considered a repetition is not permitted problem, as each person can only occupy one space. I will need to increment down by 1.
$\underline{5}^{*} \underline{4} * \underline{3} * \underline{2} * \underline{1}=120$
Answer: 120 ways
B) How many ways can they line up if Larry insists on being in the middle?

I need to form 5 spaces again, one for each person. Larry's position is pre-determined. I will begin by putting a 1 above Larry's spot in line, as there is only one choice for that position.


Next I will fill in the rest of the numbers. The first person can be any of the other 4 and I will increment down by 1 person as the same person can only hold one spot in line.
$\underline{4} * \underline{3} * \underline{1} * \underline{2} * \underline{1}=24$

Answer: 24 ways
C) If Sue insists on being first and Bob has to be last, how many ways?

I set up my 5 spots (one for each person). I put a 1 above Sue and Bob's position as there is only one person that can stand in those spots.


There are 3 people that can line up second, and I will increment down by 1 again.
$\underline{1}^{*} \underline{3} * \underline{2} * \underline{1} * \underline{1}=6$
Answer: 6 ways

I need to introduce a new word - (permutation) and a new formula. This formula gives us another way to solve problems that can be solved using the Counting Principle. Most of the problems in this section can be solved using two methods:

1) Counting Principle (creating a string of numbers and multiplying)
2) Permutation formula (putting numbers in a formula)

The permutation formula is quite a bit trickier to use when solving the types of problems in this section. I will solve a few problems both ways. That is with both the permutation formula and using the Counting Principle. You won't be responsible for solving problems with the permutation formula on the test.

Permutations: A permutation is any ordered arrangement of a given set of objects.
Permutations that involve ordering $r$ of $n$ distinct objects obey the formula:
${ }_{n} P_{r}=\frac{n!}{(n-r)!}$
We need to be familiar with the factorial symbol "!" to use the permutation formula.

Where $n!=n(n-1)(n-2) \ldots .(1)$

For example 6! $=6^{*} 5^{*} 4^{*} 3^{*} 2^{*} 1=720$
And $3!=3^{*} 2^{*} 1=6$
Note: 0!
By definition equals 1. $0!=1$

Example: The letters S, T, A, R, P are placed in a hat. A letter is removed one at a time from the hat and is not replaced. This process is repeated until all of the letters have been removed. How many different ways can this be done?

This problem can be solved using the permutation formula above as this is an ordered arrangement of 5 of 5 letters. We also can use the counting principle and make 5 lines, put numbers above the lines and multiply. I prefer the counting principle, but both methods will work.

Solving using the permutation formula:
$n=5$ and $r=5$ (" $n$ " represents the number of letters in the hat, and " $r$ " represents the total number of letters I am removing.)
${ }_{n} P_{r}={ }_{5} P_{5}=\frac{5!}{(5-5)!}=\frac{5 * 4 * 3 * 2 * 1}{0!}=\frac{120}{1}$

Answer: 120 ways

This problem can also be solved using the Counting Principle. To do this I make 5 lines, one for each letter. There are 5 letters to choose for the first letter. I am assuming that a letter can't be used more than once. I will increment down by 1 each time.
$\bar{L} \bar{L} \bar{L}<$
$\underline{5} * \underline{4} * \underline{3} * \underline{2} * \underline{1}=120$

Answer: 120 ways

Example: The letters S, T, A, R, P are placed in a hat. A letter is removed one at a time from the hat and is not replaced. This process is repeated 3 times. That is 3 letters are removed and 2 are left in the hat. How many ways can this be done?

This problem can be solved using the permutation formula above as this is an ordered arrangement of 3 of 5 letters. We also can use the counting principle. I prefer the counting principle, but both methods will work.

Solving using the permutation formula:
$n=5$ and $r=3$ (" $n$ " represents the number of letters in the hat, and " $r$ " represents the number of letters I am removing.)
${ }_{n} \mathrm{P}_{\mathrm{r}}={ }_{5} \mathrm{P}_{5}=\frac{5!}{(5-3)!}=\frac{5 * 4 * 3 * 2 * 1}{2!}=\frac{120}{2}$
Answer: 60 ways

This problem can also be solved using the Counting Principle. To do this I make 5 lines, one for each letter. There are 5 letters to choose for the first letter. I am assuming that a letter can't be used more than once. I will increment down by 1 each time.

$$
1^{\text {st }} \text { letter } \quad 2^{\text {nd }} \quad 3 \text { rd }
$$

$\underline{5} * \underline{4} * \underline{3}=60$

Answer: 60 ways
(Notice the permutation formula takes me to the same answer.)

Example: A club consisting of eight members wishes to randomly select a president, vice-president and secretary, how many different arrangements are possible?

Solved using the permutation formula as it is an ordered arrangement of 3 out of 8 objects:

$$
{ }_{8} P_{3}=\quad \frac{8!}{(8-3)!}=\frac{8!}{5!}=\frac{8 * 7 * 6 * 5 * 4 * 3 * 2 * 1}{5 * 4 * 3 * 2 * 1}=8 * 7 * 6=336
$$

Solved using the Counting Principle (I solve this as a without repetition problem as I assume one person can't hold two jobs.)
$\overline{\text { President }} \overline{\text { VP }} \overline{\text { Secretary }}$
$\underline{8}^{*} \underline{7}^{*} \underline{6}=336$
Answer: 336 ways

Permutations of Duplicate Objects.

The Counting Principle only works when each object is distinct. It doesn't work when there are duplicate objects. It over counts when there are duplicate objects.

Let me try to explain this by using an example.
Example: How many ways are there to order the letters $\mathrm{a}, \mathrm{a}, \mathrm{b}$ ?

If you use the Counting Principle you would make three slots.


You would do this next.

3 * 2 * $1=6$ ways.

Answer: 6 ways (Answer should be 3.) The Counting Principle over counts if there are duplicate objects. In our problem there are 2 "a's" and the Counting Principle will not work.

The answer should be 3. It is easy to list the three ways.
aab, aba, baa

There is no other way to arrange these letters that looks different.

I am stuck using a formula when each object is not unique.
Here is the formula:

The number of distinct permutations of $n$ objects where $n_{1}$ of the objects are identical, $\mathrm{n}_{2}$ of the objects are identical... $\mathrm{n}_{\mathrm{r}}$ of the objects are identical is found by the formula...

$$
\frac{n!}{n_{1}!n_{2}!n_{3}!\cdots n_{r}!}
$$

This is an impressive formula. Basically the numerator is the total number of objects factorial and the denominator contains a factorial for each duplicated object. This formula will be easy to use once we do a few examples.

Here is how the $a, a, b$ example will be solved using the formula:
The numerator needs a 3 ! as we are arranging 3 letters.
The denominator needs a 2 ! as there are 2 a's.
$\frac{3!}{2!}=\frac{3 * 2 * 1}{2 * 1}=3$
Answer: 3 ways

Example: In how many ways can the letters in the word "TALLAHASSEE" be arranged? (There really isn't a nice way to solve this using the counting principle because of the duplicate letters and we are stuck with the formula approach)

Work: Of the 11 letters, 3 are A's, 2 are L's, 2 are S's and 2 are E's.
The numerator will get 11! as there are 11 letters.
Each repeated letter gets a factorial in the denominator. The denominator will have 3 ! for the 3 A's, 2 ! for the 2 L's, 2 ! for the $2 \mathrm{~S}^{\prime} \mathrm{s}, 2$ ! for the 2 E's,

$$
\frac{11!}{3!2!2!2!}=\frac{11 * 10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1}{3 * 2 * 1 * 2 * 1 * 2 * 1 * 2 * 1}=831,600
$$

Answer: 831,600 ways

Example: In how many ways can the letters in the word "starburst" be arranged?
Work: Of the 9 letters, 2 are R's, 2 are S's and 2 are T's.

The numerator will get 9! as there are 9 letters.
Each repeated letter gets a factorial in the denominator. The denominator will have 2 ! for the 2 R's, 2 ! for the $2 \mathrm{~S}^{\prime} \mathrm{s}, 2$ ! for the $2 \mathrm{~T}^{\prime} \mathrm{s}$,

$$
\frac{9!}{2!* 2!* 2!} \frac{9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1}{2 * 1 * 2 * 1 * 2 * 1 * 2 * 1}=45360
$$

Answer: 45,360 ways

Homework:

1) The daily double at most racetracks consists of selecting the winning horse in both the first and the second race. If the first race has 7 entries and the second has 8 entries, how many daily double tickets mush you purchase to guarantee a win?
2) If a club consists of 10 members, how many different arrangements of president, vice-president and secretary are possible?
3) A lock combination consists of three numbers between 0 and 24 inclusive. How many combinations are possible, assume repetition is allowed?
4) A lock combination consists of three numbers between 0 and 24 inclusive. How many combinations are possible, assume repetition is NOT allowed?
5) A social security number consists of nine digits. How many different social security numbers are possible if repetition of digits is NOT permitted?
6) A social security number consists of nine digits. How many different social security numbers are possible if repetition of digits is permitted?
7) A class ring has three stone options (emerald, ruby or sapphire), two metal options (gold and silver). How many different class rings are possible?
8) Anthony has 6 dress shirts and 4 ties. How many shirt and tie outfits can he make?
9) How many different sandwiches can be made with 2 choices of bread, 3 choices of toppings and 2 choices of meat?
10) A batting order must be created from the 9 starters on a baseball team. How many batting orders are possible?
11) A batting order must be created from the 9 starters on a baseball team. How many batting orders are possible if the pitcher must bat last?
12) A batting order must be created from the 9 starters on a baseball team. How many batting orders are possible is a specific person must hit $4^{\text {th }}$ ?
13) A batting order must be created from the 9 starters on a baseball team. How many batting orders are possible if the first 4 batters are already assigned?
14) A multiple choice test has 5 questions and each question has 4 choices. How many ways are there to answer the questions?
15) A multiple choice test has 5 questions and each question has 4 choices. How many ways are there to answer the questions if you know the answer to the first question?
16) A multiple choice test has 5 questions and each question has 4 choices. How many ways are there to answer the questions if you know the answer to the first two questions?

Homework 17-20: an identification code is to consist of two letters followed by 4 digits. How many different codes are possible if
17) Repetition is permitted
18) The first letter must be $A, B, C$ or $D$ and repetition is not permitted?
19) Repetition of numbers is permitted, but repetition of letters is not permitted?
20) Repetition is not permitted?

Homework \#21-24: a license plate is to consist of three digits followed by two uppercase letters. Determine the number of different license plates possible if
21) Repetition of numbers and letters is not permitted
22) Repetition of numbers and letters is permitted
23) The first digit cannot be a zero, and repetition is not permitted
24) The last digit must be even and repetition is permitted

Homework \#25-30: Use the permutation of distinct objects formula $\frac{n!}{n_{1}!n_{2}!n_{3}!\cdot n_{r}!}$ to solve the following
25) How many ways can the letters of Mississippi be arranged?
26) How many ways can the letters of Pennsylvania be arranged?
27) How many ways can the letters $x, x, x, y, y, z, z, z, z$ be arranged?
28) How many ways can the letters $a, b, b, b, c, c, c, d, d, e$ be arranged?
29) How many ways can the letters of abracadabra be arranged?
30) How many ways can the letters of Diamondbacks be arranged?

Answers:

1) $7 * 8=56$ 3) $25 * 25 * 25=15,625$ 5) $10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2=3,628,800$ ways
2) $3 * 2=6$ 9) $2 * 3 * 2=12$ 11) $8 * 7 * 6 * 5 * 4 * 3 * 2 * 1 * 1=40,320$
3) $1 * 1 * 1 * 1 * 5 * 4 * 3 * 2 * 1=120$
4) $1 * 4 * 4 * 4 * 4=256$ 17) $26 * 26 * 10 * 10 * 10 * 10=6,760,000$
$\begin{array}{lll}\text { 19) } 26 * 25 * 10 * 10 * 10 * 10=6,500,000 & 21) \\ 10 * 9 * 8 * 26 * 25=468,000\end{array}$
5) $9 * 9 * 8 * 26 * 25=421,200$ 25) $\frac{11!}{4!* 4!* 2!}=34,650$
6) $\frac{9!}{3!* 2!* 4!}=1260 \quad$ 29) $\frac{11!}{5!* 2!* 2!}=83,160$
