Section 4.7: Conditional Probability

Conditional Probability is the probability that an event will occur, ***GIVEN*** that another event has occurred.

This is a more mathematically elegant definition:

Conditional Probability is the probability of event two (E_2) happening given that event one (E_1) has happened.

This is the symbolism that is used in most conditional probability problems: $P(E_2|E_1)$

The symbol $P(E_2|E_1)$ asks us to find the probability that event E_2 occurs given event E_1 has occurred. (The vertical line stands for the words “given that”.)

There are two formulas we will use to solve conditional probability problems. Here is the first conditional probability formula:

$$P(E_2|E_1) = \frac{number\ of\ occurences\ of\ the\ desired\ event\ in\ the\ reduced\ sample\ space}{number\ of\ items\ in\ the\ reduced\ sample\ spae}$$

We will use this formula when it is relatively painless to create a sample space for the probability experiment defined in the problem.

Example: A single six sided dice is rolled one time. Determine the probability that a 2 was rolled, ***GIVEN*** an even number has been rolled.

This is a conditional probability question. The words ***given that*** clue me into the fact that this is a conditional probability problem. The condition is that I know that the dice came up an even number when it was rolled.

The sample space for this experiment is: S = {1,2,3,4,5,6}

The sample space was easy to generate, so I will use the conditional probability formula that contains the words sample space. This is the only formula introduced so far in this section.

This question can be represented symbolically as follows:

$P(2\ is\ rolled\ |\ even\ number\ has\ been\ rolled)$

$P(2\ is\ rolled\ |\ even\ number\ has\ been\ rolled) = \frac{The\ number\ of\ 2\text{'s in the reducted sample space}}{The\ number\ of\ elements\ in\ the\ reduced\ sample\ space}$

To solve this problem I first need to create a reduced sample space as it will help me find the denominator in my answer.

The reduced sample space will only contain even numbers, as I have been told an even number has been rolled.

Reduced sample space = {2,4,6}

The denominator for our answer will be 3 as there are 3 items in the reduced sample space.

The numerator for our answer will be any occurrence in the reduced sample space that matches the requirement. The dice is required to be a 2. There is one 2 in the reduced sample space so the numerator of my answer is 1.

P(2 is rolled given the dice is even) $= \frac{The\ number\ of\ 2\text{'s in the reducted sample space}}{the\ number\ of\ elements\ in\ the\ reduced\ sample\ space} = \frac{1}{3}$

Answer: 1/2

Example: A family has two children. Assuming that boys and girls are equally likely, determine the probability that the family has…

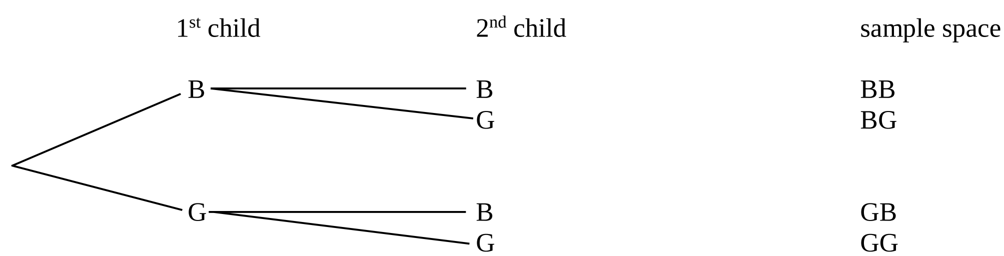
A) One boy and one girl ***GIVEN*** the first child is a boy.

B) Two girls ***GIVEN*** that at least one is a girl

C) Two girls ***GIVEN*** that the older one is a girl

Again, it is not too hard to create a sample space. I will create a sample space and solve this more or less like I solved that last example.

Solution: Draw a tree diagram to help "visualize" this.



A) P(one boy and one girl ***GIVEN*** the first child is a boy) =

First I create a reduced sample space by selecting all elements of the original sample space in which the first child is a boy.

Reduced sample space = {BB BG}

The denominator for my answer will be 2 as there are 2 elements in the reduced sample space.

The numerator for my answer will be the number of occurrences in the reduced sample space that have one boy and one girl. There is one such occurrence the BG. The numerator will be 1.

P(one boy and one girl given the first child is a boy) = $\frac{number\ of\ elements\ in\ reduced\ sample\ space\ with\ one\ boy\ and\ one\ girl}{number\ of\ elements\ in\ the\ reduced\ sample\ space}$

Answer: P(one boy and one girl given the first child is a boy) = $\frac{1}{2}$

B) P (two girls ***GIVEN*** at least one girl) =

First I need to create a reduced sample space by selecting all of the elements of the original sample space that have at least one girl.

Reduced sample space = {BG GB GG}

The denominator of my answer will be 3.

The numerator of my answer will be the number of occurrences in the reduced sample space that have the desired characteristic of two girls. There is one such occurrence, the GG. The numerator will be 1.

P (two girls given at least one girl) = $\frac{number\ of\ elements\ in\ reduced\ sample\ space\ with\ two\ girls}{number\ of\ elements\ in\ the\ reduced\ sample\ space}$

Answer: P (two girls given at least one girl) = $\frac{1}{3}$

C) $P\left(both\ girls \middle| oldest\ is\ a\ girl\right) =$

First I need to create a reduced sample space by selecting all of the elements of the original sample space that have the oldest child as a girl.

Reduced sample space = {BG GG}

The denominator of my answer will be 2.

The numerator of my answer number of occurrences in the reduced sample space that have the desired characteristic of both girls. There is one such occurrence, the GG. The numerator will be 1.

P (two girls given at least one girl) = $\frac{number\ of\ elements\ in\ reduced\ sample\ space\ with\ two\ girls}{number\ of\ elements\ in\ the\ reduced\ sample\ space}$

Answer: $P\left(both\ girls \middle| oldest\ is\ a\ girl\right) = \frac{1}{2}$

Homework #1 – 8: Create a reduced sample space and find the requested probabilities. A family has three children. Assuming that boys and girls are equally likely, determine the probability that the family has… (Write your answer as a reduced fraction.)

1) Two girls given the first child is a girl
2) Two girls given the first child is a boy
3) Less than 2 girls given the first child is a girl
4) More than 2 girls given the first child is a girl.
5) Two boys given the middle child is a boy.
6) 1 boy given the middle child is a girl
7) 1 girl given the oldest child is a girl
8) Two girls given the oldest child is a girl.

Homework #9 – 16: Create a reduced sample space and find the requested probabilities. A pair of dice is rolled and the sum of the dice is recorded, determine the probability that: (Write your answer as a reduced fraction.)

9) The sum is greater than 5 given the first dice is a 4.
10) The sum is greater than 9 given the second dice is a 6.
11) The sum is even given the second dice is a 4
12) The sum is odd given the first dice is a 3
13) A double is rolled given neither dice is a 4
14) A double is not rolled given the sum is greater than 10
15) The sum is less than 3 given the first dice is a 2
16) The sum is greater than 10 given the first dice is a 3

Homework #17-24: Create a reduced sample space and find the requested probabilities. One card is selected from a deck of cards find the probability that: (Write your answer as a reduced fraction.)

17) The card is heart given it is not black
18) The card is a spade given it is red
19) The card is a four given that it is black
20) The card is a queen given that it is a face card
21) The card is not a seven given it is between 3 and 8 inclusive
22) The card is not a seven given it is between 3 and 8 exclusive
23) The card is not red given it is a queen
24) The card is not black given it is a two

Homework: #25 – 28 Assume that a hat contains 4 bills: a \$1 bill, a \$5 bill, a \$10 bill and a \$20 bill. Two bills are to be selected at random with replacement. Construct a sample space, and find the probability that: (Write your answer as a reduced fraction.)

25) Both bills are \$1 bills if given the first selected is a \$1 bill
26) Both bills have a value greater than a \$5 bill given the second bill is a \$10 bill.
27) The second bill is a five given the first bill is a \$5.
28) The first bill is a five given the second bill is a \$5.

Often it is not practical to create a reduced sample space to answer conditional probability questions. We can use a different conditional probability formula to solve problems when it is not practical to create a reduced sample space. This formula is equivalent to the formula we have already used it just looks a little different.

For and two events E_1 and E_2

$$P(E_2|E_1) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{number\ of\ ways\ event\ 1\ and\ event\ 2\ occur\ together}{number\ of\ ways\ event\ 2\ occurs}$$

Example: Two hundred patients who had either hip surgery or knee surgery were asked whether they were satisfied or dissatisfied regarding the result of their surgery. The following table summarizes their response.

Surgery	Satisfied	Dissatisfied	Total
Knee	70	25	95
Hip	90	15	105
Total	160	40	200

A) If one person from the 200 patients is selected at random, determine the probability that the person was satisfied ***GIVEN*** that the person had knee surgery.

This is how the question might be written symbolically:

P(satisfied given person had knee surgery)=

This is a conditional probability problem as it contains the words "given that".

To use the above formula I need to create two events. E_1 and E_2. For our problem:

E_1 = Satisfied.

E_2 = Knee surgery.

Next I need to find:
$n(E_1 \cap E_2)$ This is the number of people that had knee surgery and were satisfied.

$n(E_1 \cap E_2)$ = 70

Next I need to find $n(E_2)$ which is the number of people that had knee surgery.

$n(E_2) = 95$

We put the numbers in n the formula like this

P(satisfied given person had knee surgery)= $P(E_2|E_1) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{70}{95} = \frac{14}{19} = 73.7\%$

Answer: 73.7% chance

B) Was dissatisfied ***GIVEN*** they had hip surgery

First create two events.

E_1 = people that were dissatisfied.

E_2 = people that had hip surgery

Next I need to find:
$n(E_1 \cap E_2)$ This is the number of people that had hip surgery and were dissatisfied.

$n(E_1 \cap E_2)$ = 70

Next I need to find $n(E_2)$ which is the number of people that had hip surgery.

$n(E_2)$ = 95

We put the numbers in n the formula like this

P(dissatisfied given person had hip surgery)= $P(E_2|E_1) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{15}{105} = \frac{1}{7} = 14.3\%$

Answer: 14.3% chance

C) Had knee surgery ***GIVEN*** they were dissatisfied = $P(E_2|E_1) = \frac{n(E_1 \cap E_2)}{n(E_2)}$

First create two events:

E_1 = people that had knee surgery.

E_2 = people that were dissatisfied.

Next I need to find:

$n(E_1 \cap E_2)$ This is the number of people that had knee surgery and were dissatisfied.

$n(E_1 \cap E_2)$ = 25

Next I need to find $n(E_2)$ which is the number of people that were dissatisfied.

$n(E_2) = 40$

We put the numbers in n the formula like this

P(had knee surgery given they were dissatisfied) $P(E_2|E_1) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{25}{40} = \frac{5}{8} = 62.5\%$

Answer: 62.5% chance

Homework #29 – 32: At a zoo, a sampling of children was asked if the zoo were to get one additional animal, would they prefer a lion or an elephant. The results of the survey follow: (Write your answer as a reduced fraction.)

	Lion	Elephant	Total
Boys	90	110	200
Girls	75	85	160
Total	165	195	360

If one child who was in the survey is selected at random, find the probability that

29) The child selected the lion, given the child is a girl.
30) The child is a boy, given the child preferred the elephant.
31) The child selected is a girl, given that the child preferred the lion.
32) The child preferred the elephant given they are a girl.

Homework #33-36: A quality control inspector, is checking a sample of light bulbs for defects. The following table summarizes her findings. (Write your answer as a reduced fraction.)

Wattage	Good	Defective	Total
20	80	15	95
50	100	5	105
100	120	10	130
Total	300	30	330

If one of these light bulbs is selected at random, find the probability that the light bulb is

33) Good given it is 100 watts
34) Defective given it is 50 watts
35) 100 watts given it is good
36) 50 watts given it is defective

Homework #37 – 40: A survey of students to determine if they had a pierced ear was given. The results are summarized in the table below. (Write your answer as a reduced fraction.)

	Pierced	Not pierced	Total
Male	36	144	180
Female	288	32	320
Total	324	176	500

If one person is selected at random find the probability that

37) Female given they are pierced
38) Male given they are not pierced
39) Not pierced given they are female
40) Not pierced given they are male

Answers:
1) 2/4 = ½ 3) 1/4 5) 2/4 = ½ 7) ¼ 9) 5/6 11) 3/6 = ½ 13) 5/25 = 1/5 15) 0/6 = 0

17) 13/26 = ½ 19) 2/26 = 1/13 21) 20/24 = 5/6 23) 2/4 = ½ 25) ¼ 27) ¼ 29) 75/160 = 15/32

31) 75/165 5/11 33) 120/130 = 12/13 35) 120/300 = 2/5 37) 288/324 = 8/9 39) 32/320 = 1/10